B.Sc. Part III 7th Paper

Diff. equations

Second order Equations with variable coefficients

Linear equation of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q, R are functions of x alone.

Important Formulae

1. If $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$ or $a^2 + Pa + Q = 0$

   then $y = e^{ax}$ is a part of CF.

2. If $1 + P + Q = 0 \Rightarrow y = e^{x}$ is a part of CF.
3. If $1 - P + Q = 0 \Rightarrow y = e^{-x}$ is a part of CF.
4. If $m(m-1) + Pmx + Qx^2 = 0 \Rightarrow y = x^m$ is a part of CF
5. If $P + Qx = 0 \Rightarrow y = x$ is a part of CF.
6. If $2 + 2Px + Qx^2 = 0 \Rightarrow y = x^2$ is a part of CF.

1. Prove that $y = \sin x$ is a part of CF of the equation

$(\sin x - x\cos x)y'' - x\sin x\, y' + y\sin x = 0.$

Soln. The given equation

$(\sin x - x\cos x)y'' - x\sin x\, y' + y\sin x = 0$ ——— (1)

$\because y = \sin x \Rightarrow y' = \cos x$ and $y'' = -\sin x$

Putting these values in the LHS of (1),

we get

$LHS = (\sin x - x\cos x).(-\sin x) - x\sin x.\cos x + \sin x.\sin x$

$= \cancel{-\sin^2 x} + \cancel{x\sin x\cos x} - \cancel{x\sin x\cos x} + \cancel{\sin^2 x}$

$= 0 = RHS.$

Hence $y = \sin x$ is a part of CF of (1).

nsid Q. Solve $xy'' - (2x-1)y' + (x-1)y = 0$

or

Solve by reducing the order $xy'' - (2x-1)y' + (x-1)y = 0$ given that e^x is one integral part.

Soln. The given equation

$$xy'' - (2x-1)y' + (x-1)y = 0 \quad \text{---(1)}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \left(\frac{2x-1}{x}\right)\frac{dy}{dx} + \left(\frac{x-1}{x}\right)y = 0 \quad \text{---(1)}$$

This is of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

Here $P = -\frac{2x-1}{x} = -\left(2 - \frac{1}{x}\right) = -2 + \frac{1}{x}$

$Q = \frac{x-1}{x} = 1 - \frac{1}{x}$

Now, $1 + P + Q = 1 - 2 + \frac{1}{x} + 1 - \frac{1}{x} = 0$

$\Rightarrow y = e^x$ is a part of CF of the eq (1).

Let the complete solution of (1) be

$$y = ve^x \quad \text{---(2)}$$

$$\Rightarrow \frac{dy}{dx} = e^x\frac{dv}{dx} + ve^x = e^x\left(v + \frac{dv}{dx}\right) \quad \text{---(3)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x\frac{d^2v}{dx^2} + e^x\frac{dv}{dx} + e^x\frac{dv}{dx} + ve^x = e^x\left(\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v\right) \quad \text{---(4)}$$

using (2), (3) and (4) in (1), we get

$$e^x\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right)-\left(2-\frac{1}{x}\right)e^x\left(v+\frac{dv}{dx}\right)+\left(1-\frac{1}{x}\right)ve^x=0$$

$$\Rightarrow e^x\left(\frac{d^2v}{dx^2}\right)+e^x\frac{dv}{dx}\left[2-\left(2-\frac{1}{x}\right)\right]+ve^x\left[1-\left(2-\frac{1}{x}\right)+\left(1-\frac{1}{x}\right)\right]=0$$

$$\Rightarrow e^x\frac{d^2v}{dx^2}+e^x\frac{1}{x}\frac{dv}{dx}=0 \Rightarrow e^x\left(\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}\right)=0$$

$$\Rightarrow \frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}=0, \quad \text{as } e^x\neq 0 \quad (5)$$

Put $\frac{dv}{dx}=q \Rightarrow \frac{d^2v}{dx^2}=\frac{dq}{dx}$

$$(5)\Rightarrow \frac{dq}{dx}+\frac{1}{x}q=0$$

$$\Rightarrow \frac{dq}{q}+\frac{dx}{x}=0. \text{ Integrating we get}$$

$$\log q+\log x=\log k \Rightarrow qx=k \Rightarrow q=\frac{k}{x}$$

$$\Rightarrow \frac{dv}{dx}=\frac{k}{x} \Rightarrow dv=k\frac{dx}{x}$$

$$\Rightarrow \int dv=k\int\frac{dx}{x} \Rightarrow v=k\log x+k_1 \quad (6)$$

where k and k_1 are constants

From (2), $y=ve^x$

$$\Rightarrow y=(k\log x+k_1)e^x \quad \text{using (6)}$$

This is the complete solution.